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A pressure-correction algorithm with Mach-uniform efficiency and accuracy

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SUMMARY

We present a collocated Mach-uniform pressure-correction method. By using a low Mach adapted AUSM+ flux for the spatial discretization, we reach Mach-uniform accuracy. Mach-uniform efficiency is obtained by a pressure-correction equation based on the energy equation. Furthermore, we take heat conduction into account, which as far as we know, has never been done before in the context of Mach-uniform pressure-correction methods. An explicit treatment of the conduction terms results in a diffusive limit on the time step. To avoid this, a coupled solution of the energy equation and the continuity equation is needed. The results for both the adiabatic and the non-adiabatic algorithms are in full accordance with the developed theory. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: algorithm; pressure-correction; Mach-uniform; adiabatic; non-adiabatic

1. INTRODUCTION

Mach-uniform algorithms are an indispensible tool in numerous flow situations [1]. For years, the CFD-world has been searching for the ideal algorithm that can handle any level of Mach number. With preconditioning, the originally high speed density-based algorithms were extended towards the low Mach number regime [2]. The solution technique is then coupled, involving the solution of large systems. As a segregated algorithm, the pressure-correction method is a well-established technique for incompressible flow [3]. However, the extension towards the higher Mach numbers requires special attention. Several attempts have been done to develop compressible pressure-correction methods [4–12], but only a few are really Mach-uniform, meaning that they reach a good accuracy and convergence over the whole

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Mach number range [1, 11, 12]. In this paper, we present a Mach-uniform, collocated pressure-correction method. Furthermore, we will take into account heat conduction, which, as far as we know, has never been done before in the context of Mach-uniform pressure-correction algorithms. In the present paper we explain the basic idea behind the algorithm. For a detailed description of the mathematical background and implementation, we refer to a future paper [13].

2. GOVERNING EQUATIONS AND FINITE VOLUME METHOD

We consider a one-dimensional inviscid flow of a perfect gas. The governing Navier–Stokes equations are the continuity, momentum and energy equation. The conductive heat flux is expressed by Fourier's law and is discretized centrally. The equations are nondimensionalized by choosing reference quantities for pressure, temperature and length scale. A collocated vertex-centred finite volume method and an Euler implicit time integration scheme are applied to discretize the Navier–Stokes equations.

3. CONSTRUCTION OF A MACH-UNIFORM PRESSURE-CORRECTION METHOD

3.1. Mach-uniform accuracy

We use the advection upwind splitting method (AUSM+) for the spatial discretization [14], where the transported quantities are first-order upwinded. At high Mach numbers the AUSM-flux performs very well. At low Mach numbers special measures have to be taken with respect to the scaling and decoupling problem [15].

3.2. Mach-uniform efficiency

The low Mach stiffness problem causes a breakdown of convergence at low Mach numbers. This can be remedied by treating the terms which contain acoustic information implicitly [16]. The quasi-linear Euler equations, learn that the acoustic information, transported with velocity $u \pm c$, is given by $\delta u \pm \delta p/\rho c$. When the continuity equation is removed from the quasi-linear set, the quantities transported at velocity $u \pm c$ are unchanged. Therefore, the reduced system still contains all acoustic information. We conclude that all acoustic information is contained in the momentum and energy equations. Therefore, corrections will be introduced into the latter two equations, leading to a pressure-correction equation that is based on the energy equation.

4. IMPLEMENTATION FOR ADIABATIC FLOW

We start with an adiabatic flow, i.e. there is no heat conduction. For this case, the presented method is in essence the same as the one in References [11, 12]. The method is conservative. The first step in the algorithm is a predictor step for density ρ^* and momentum $(\rho u)^*$ from the continuity and momentum equation, respectively. Density is updated by this value. The field thus derived, does not fulfil the energy equation. Therefore, corrections for pressure p'

and momentum $(\rho u)'$ are introduced, $p^{n+1} = p^n + p'$, $(\rho u)^{n+1} = (\rho u)^* + (\rho u)'$, with n and $n+1$, respectively, the old and new time level. The momentum corrections are related to the pressure corrections through the momentum equation. Every implicit term in the energy equation is written as a function of pressure corrections. For example, the total enthalpy flux is written as

$$(\rho H u)^{n+1}_{i+1/2} = (\rho H)^*_i u^*_{i+1/2} + H^*_i (\rho u)'_{i+1/2} + (\rho H)'_{i+1/2} u^*_{i+1/2} \tag{1}$$

$$(\rho H)'_{i+1/2} = \frac{\gamma}{\gamma - 1} p'_i + u^*_{i+1/2} (\rho u)'_{i+1/2} \tag{2}$$

This yields a pressure-correction equation, from which pressure is corrected. After this, momentum and velocity are corrected. The equation of state is used to update temperature.

5. IMPLEMENTATION FOR NON-ADIABATIC FLOW

Due to the heat conduction terms, temperature now appears into the energy equation, which we used so far as pressure determining. We present two different approaches to cope with this.

5.1. Pressure-correction method with explicit temperature update

We can simply leave the heat transfer terms into the RHS of the energy equation and calculate them by means of old temperatures values. Therefore, extra terms are added to the RHS of the pressure-correction equation. The rest of the procedure remains unchanged. We refer to this method as *explicit*, because the temperature terms in the energy equation are treated in an explicit way. Consequently, we expect a time step limit due to the diffusive terms. The latter is determined by the Von Neumann number, $Ne = \kappa \Delta t / \Delta x^2$, with κ the conduction coefficient, Δx the cell length and Δt the time step.

5.2. Coupled pressure and temperature correction method

To remove the diffusive time step limit, the temperature terms in the energy equation have to be treated implicitly. Therefore, we introduce temperature corrections T' into the RHS of the energy equation. The LHS is treated as for the non-adiabatic case, thus introducing pressure corrections. The discretized energy equation therefore becomes an equation for both pressure and temperature corrections. A second equation containing both pressure and temperature corrections, is derived from the continuity equation. The density is expanded as

$$\rho^{n+1}_i = \rho^*_i + \left.\frac{\partial \rho}{\partial p}\right|^*_{T=cte,\, i} p'_i + \left.\frac{\partial \rho}{\partial T}\right|^*_{p=cte,\, i} T'_i \tag{3}$$

and the momentum flux is again related to pressure corrections through the momentum equation. This results in a second equation for pressure and temperature corrections. The two correction equations are solved in a coupled way. We therefore refer to this method as *coupled correction method*. The predictor step is the same as in the adiabatic algorithm. From the coupled solution of p' and T', pressure and temperature are updated. Density is updated through (3). All systems are solved with a direct solver, though the positive structure allows for a more optimized solver.

6. RESULTS

As a test case, we take a converging–diverging one-dimensional nozzle. These simple tests will show clearly the basic features of the described Mach-uniform algorithm.

6.1. Adiabatic flow

First we test the adiabatic implementation of Section 4. Both low and high speed flows are considered.

6.1.1. Low speed flow. We consider a subsonic nozzle flow with a throat Mach number M_t of 10^{-3}. The time step $\tau = \Delta t/\Delta x$ is calculated from a chosen convective CFL-number, i.e. $\tau = \mathrm{CFL_u}/\max(u)$. We remark that this corresponds with an acoustic CFL number that is about 1000 times higher. The $\mathrm{CFL_u}$ number could be taken arbitrarily high; there is no stability limit. Figures 1(a) and (b) show the results for pressure and Mach number, computed at a $\mathrm{CFL_u}$ number of 10. Figure 1(c) shows the convergence plots for computations at $\mathrm{CFL_u}$ numbers of 1 and 10. The Mach-uniform algorithm shows an excellent convergence rate.

In the literature, we found two examples of pressure-correction methods which also use an energy equation for the pressure corrections [1, 11, 12, 17]. Based on a low Mach number perturbation analysis [1, 18], a parameter M_r, which indicates the Mach number level, is introduced. Furthermore, the pressure is split in a zeroth- and second-order part. This shows clearly what happens in the low Mach number limit, but it is not essential in order to obtain Mach-uniform efficiency. In our algorithm, we do not use the parameter nor the explicit pressure splitting: the algorithm reduces *automatically* to the right low Mach number limit. Finally, our algorithm is formulated on a collocated grid, while in the cited examples a staggered grid is used.

Figure 1(c) also shows the convergence plots for a pressure-correction algorithm that uses the continuity equation to construct the pressure-correction equation. The computations could only be made stable under a severe underrelaxation, and it has a very bad convergence rate

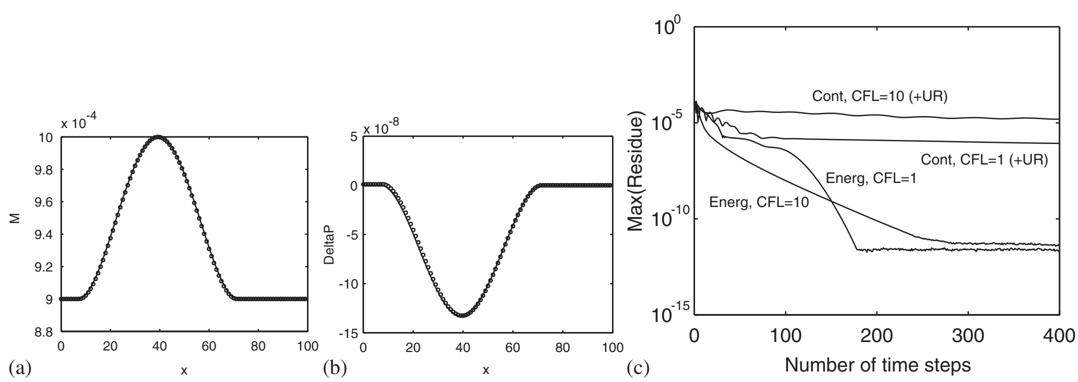

Figure 1. Subsonic nozzle flow, $M_t = 0.001$: (a) Mach number; (b) relative pressure ($p - p_{out}$). Symbols: Mach-uniform algorithm, $\mathrm{CFL_u} = 10$. Solid line: analytic solution; and (c) convergence plot, $\mathrm{CFL_u} = 1$ and 10. Energ: Mach-uniform algorithm. Cont: equivalent algorithm based on the continuity equation (computation with underrelaxation (UR)).

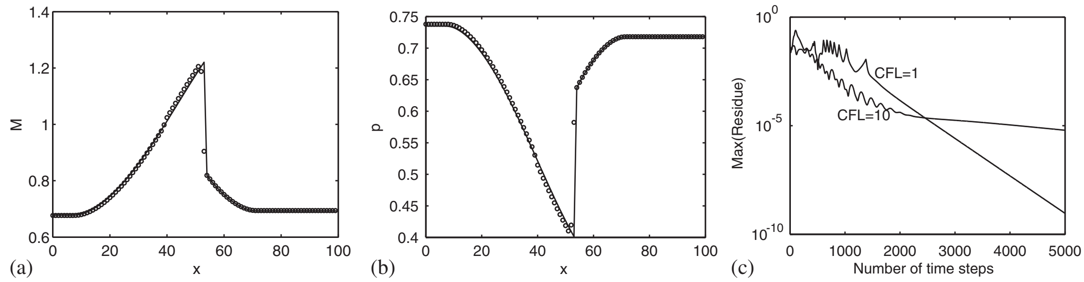

Figure 2. Transonic nozzle: (a) Mach number; (b) pressure distribution. Symbols: Mach-uniform algorithm, $CFL_u = 1$. Solid line: analytic solution; and (c) convergence plot, $CFL_u = 1$ and 10.

Table I. Nozzle flow.

			Stable?		Time steps		Calc time	
Nozzle	κ	Ne	Exp	Coup	Exp	Coup	Exp	Coup
Subsonic	10^{-5}	10^{-3}	Yes	Yes	220	208	19.6	23.5
$M_t = 0.01$	0.01	1	Yes	Yes	259	224	22.3	24.5
$Ne \approx \kappa CFL_u/0.01$	0.1	10	No	Yes	—	337	—	36.8
	1	100	No	Yes	—	646	—	70.4
	10	1000	No	Yes	—	222	—	24.5
Subsonic	10^{-5}	10^{-4}	Yes	Yes	1186	1180	101.2	128.8
$M_t = 0.1$	0.1	1	Yes	Yes	1182	1176	101.1	128.4
$Ne \approx \kappa CFL_u/0.1$	1	10	No	Yes	—	1211	—	132.2
	10	100	No	Yes	—	2019	—	228.7
Transonic	10^{-5}	10^{-5}	Yes	Yes	3280	3309	290.0	372.9
$Ne \approx \kappa CFL_u/1$	1	1	Yes	Yes	3169	3200	279.9	359.0
	10	10	No	Yes	—	4850	—	549.7
	100	100	No	Yes	—	3696	—	413.3

Stability, number of time steps till convergence and calculation time, for different values of κ. Exp: explicit temperature update. Coup: Coupled correction method.

at this low Mach number. However, several examples of pressure correction methods based on the continuity equation are found in the literature [4–10].

We conclude that the Mach-uniform algorithm performs very well for this low speed flow, with regard to accuracy as well as efficiency.

6.1.2. High speed flow. Also for the case of a transonic nozzle there is no CFL-limit. Figure 2(c) shows the convergence plot for a computation at $CFL_u = 1$ and 10. The Mach number and pressure distributions are shown in Figures 2(a) and (b). We conclude that also for the high Mach numbers we reach a good convergence rate and accuracy. Clearly, the algorithm shows Mach-uniform efficiency and accuracy.

6.2. *Non-adiabatic flow*

With the time step τ derived from the convective CFL number and $\Delta x = 1$, the diffusive limit turns into a restriction for the heat conduction coefficient. Table I shows the stability results for different nozzle flows, together with the number of time steps till convergence (residue 10^{-12} for the subsonic flow and 10^{-8} for the transonic flow) and the calculation time. The results show clearly that the explicit method becomes unstable as soon as the Von Neumann number becomes higher than order unity. The coupled method, however, stays stable no matter how high the nondimensional conduction coefficient κ is taken.

7. CONCLUSION

We presented a Mach-uniform pressure-correction method. We showed how an AUSM+ flux leads to Mach-uniform accuracy, and how a pressure-correction equation based on the energy equation leads to Mach-uniform efficiency. When heat conduction is present, an explicit treatment of the conduction terms results in a diffusive limit on the time step. To escape this, a coupled solution of the energy equation and the continuity equation is needed. The tests on different nozzle flows are in full accordance with the developed theory.

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